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Structural Dynamics Analyses Testing and Correlation

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ABSTRACT

This report examines some aspects of the lack of close correlation between the predictions of analytical modeling of dynamic structures and the results of vibration tests on such structures, and suggests ways in which the correlation may be improved.

CONTENTS

CONTENTS (Continued)

SECTION I
INTRODUCTION

Designing a structure to survive a prescribed dynamic environment is most often performed nowadays by using an analytical model of the structure. Since many structures will not be subjected to their design environment prior to commissioning, it is very important, therefore, that the analytical model mimic the behavior of the physical system with reasonable accuracy. While modern analytical techniques have the capability of modeling a physical system to any desired degree of accuracy, financial or time considerations may preclude the use of highly accurate analytical models of the system. Clearly, if the cost of analytically modeling the system becomes comparable with the cost of building and testing the physical system, then it may well be that cut-and-try methods are more cost effective than analytical methods. While the aircraft and aerospace industries have used analytical modeling for decades, until recently the automotive industry found that, due to the complex double curvature of the automobile body, it was more cost efficient to use cut-and-try methods rather than analytical modeling. To compromise between cut-and-try methods and the use of highly accurate, but very expensive, analytical models, engineers are frequently willing to accept a fairly crude analytical model of the desired structure and to resort to a limited program of testing to "qualify" the analytical model. Since the engineer, unlike the mathematician or scientist, must always balance rigor against cost, an important question is, "Given that the dynamic environment is known with only limited accuracy, how accurate must the analytical model be to obtain 'adequate' predictions of the dynamic response of the structure?" This report examines a number of aspects of the problem of trying to correlate the results of dynamic testing of a structure with the analytical predictions based on rather crude modeling.

SECTION II

ANALYTICAL MODELING AND PREDICTION OF RESPONSE

Almost all structural systems are distributed parameter (continuous) systems; this is particularly true in the case of aerospace vehicles where the desire to minimize weight results in a design with mass and stiffness distributed throughout the system. The complex geometry and boundary conditions in space vehicles seldom permit exact solutions of the partial differential equations describing the dynamical behaviors of the vehicle. For this reason, finite differences, finite element, Rayleigh-Ritz, or Galerkin techniques are normally used to discretize the system and reduce the problem to that of a lumped parameter system. While these techniques differ in detail, they all have the same general properties. They attempt to approximate a space-continuous system by a discrete system having a finite number of degrees of freedom, N. A common feature of such discrete approximating schemes is that only the first M modes, $M \approx 1/3\ N$, have a reasonable chance of accurately modeling the first M modes of the continuous system. Naturally, the larger N is, the larger M may be, and the better the degree of approximation in the lower modes. The higher discrete modes are, in general, poor approximations of the continuous modes, even when N is large.

Since the stress in a continuous structure depends on the spatial derivatives of the deformations, the order N of the discrete approximating system should be large to obtain accurate approximations of the spatial derivatives; however, in many important problems in structural dynamics, the number of dynamically active modes, M, is much smaller than N. For this reason the analyst will frequently reduce the size of his model to more closely correspond to the number of active modes. This practice, unfortunately, reduces the accuracy with which stresses and forces in the structure may be determined, particularly in the case of transient motions.

A. STRUCTURAL DYNAMICS

Damping in dynamic structures is usually a parasitic effect because its exact origin and form is seldom known accurately before the structure is built. In addition, in well built structures, the damping is usually small. For these reasons it is usual to assume that the structure has viscous damping and admits classical normal modes. It will be shown that, if the damping is small and the eigenvalues well separated, this is a reasonably good approximation, at least in the lower modes.

1. Continuous Systems

To illustrate the techniques, let us restrict our attention to a relatively simple continuous structure such as a beam, a plate, or a shell that can be described by Equation 2-1:

$$\left.\begin{aligned} \rho(\underline{x})\, u_{tt} + L_1 u_t + L_2 u &= f(\underline{x}, t) \text{ on } D \\ u(\underline{x}, 0) = u_t(\underline{x}, 0) &= 0 \end{aligned}\right\} \tag{2-1}$$

with $Bu = 0$ on ∂D $\rho(\underline{x}) > 0$. Caughey and O'Kelly (Reference 2-1) have shown that Equation (2-1) admits classical normal modes if the following conditions are met:

(1) L_1 and L_2 are self-adjoint spatial operators.

(2) $\frac{1}{\rho(\underline{x})} L_1$ and $\frac{1}{\rho(\underline{x})} L_2$ commute.

(3) The boundary conditions prescribed on ∂D are compatible with the operators L_1 and L_2. Under these conditions there exists a complete set of linearly independent eigenfunctions $X_i(\underline{x})$, $i \in [1, \infty)$ such that

$$\int_D \rho(\underline{x}) X_i X_j \, d\underline{x} = \delta_{ij} \tag{2-2}$$

$$\int_D X_i L_1 X_j \, d\underline{x} = 2\omega_i \zeta_i \delta_{ij} \tag{2-3}$$

$$\int_D X_i L_2 X_j \, d\underline{x} = \omega_i^2 \delta_{ij} \tag{2-4}$$

Thus if

$$u(\underline{x}, t) = \sum_{i=1}^{\infty} y_i(t) X_i(\underline{x}) \tag{2-5}$$

Equation (2-1) reduces to

$$\ddot{y}_i + 2\omega_i \zeta_i \dot{y}_i + \omega_i^2 y_i = q_i(t)$$

where

$$q_i(t) = \int_D X(\underline{x}, t) f(\underline{x}, t)\, d\underline{x} \tag{2-6}$$

$$i \in [1, \infty]$$

For homogeneous initial data,

$$y_i(t) = \int_0^t \exp\left(-\omega_i \zeta_i (t - \tau)\right) \frac{\sin \bar{\omega}_i (t - \tau)}{\bar{\omega}_i} q_i(\tau)\, d\tau$$

$$\bar{\omega}_i = \omega_i \sqrt{1 - \zeta_i^2} \tag{2-7}$$

2. Discrete Systems

Given the system

$$M\ddot{\underline{u}} + D\dot{\underline{u}} + K\underline{u} = \underline{f}(t)$$

$$u(0) = \dot{u}(0) = 0 \quad M, D, K - N \times N \tag{2-8}$$

Caughey and O'Kelly (Reference 2-1) have shown that Equation (2-8) admits classical normal modes iff $M^{-1}K$ and $M^{-1}D$ commute.

If M, D, and K are symmetric with M positive definite and D and K at least positive semidefinite, there exists a complete set of ordinary eigenvectors $\underline{\phi}^{(i)}$ such that:

$$\underline{\phi}^{(i)T} M \underline{\phi}^{(j)} = \delta_{ij} \tag{2-9}$$

$$\underline{\phi}^{(i)T} D \underline{\phi}^{(j)} = 2\omega_i \zeta_i \delta_{ij} \tag{2-10}$$

$$\underline{\phi}^{(i)T} K \underline{\phi}^{(j)} = \omega_i^2 \delta_{ij} \tag{2-11}$$

$$i \varepsilon [1, N]$$

If we write

$$\left.\begin{aligned} \underline{u}(t) &= \mathbf{\Phi} \underline{y}(t) \\ \mathbf{\Phi} &= \left[\underline{\phi}^{(1)}, \underline{\phi}^{(2)}, \cdots \underline{\phi}^{(N)} \right] \end{aligned}\right\} \tag{2-12}$$

then

$$\left.\begin{aligned} \ddot{y}_i + 2\omega_i \zeta_i \dot{y}_i + \omega_i^2 y_i &= q_i(t) \\ y_i(0) = \dot{y}_i(0) &= 0 \end{aligned}\right\} \tag{2-13}$$

where

$$\underline{q}(t) = \mathbf{\Phi}^T \underline{f}(t)$$

For homogeneous initial data,

$$\left.\begin{aligned} y_i(t) &= \int_0^t \exp\left(-\omega_i \zeta_i (t - \tau)\right) \frac{\sin \bar{\omega}_i (t - \tau)}{\bar{\omega}_i} q_i(\tau)\, d\tau \\ \bar{\omega}_i &= \omega_i \sqrt{1 - \zeta_i^2} \\ & \qquad i \in [1, N] \end{aligned}\right\} \quad (2\text{-}14)$$

3. Excitation of Pure Modes

If in the case of Subsections II-A-1 and II-A-2 the forcing function is given by

$$f(\underline{x}, t) = \underline{\rho}(\underline{x}) X_i(\underline{x}) p(t)$$

or

$$\underline{f}(t) = M \underline{\phi}^{(i)} p(t) \quad (2\text{-}15)$$

we see that

$$q_j(t) = p(t)\delta_{ij} \quad (2\text{-}16)$$

Hence, in the case of Subsection II-A-1,

$$\left.\begin{aligned} y_j(t) &= \int_0^t h_j(t - \tau)\, p(\tau)\, d\tau\ \delta_{ij} \\ h_j(t) &= \exp\left(-\omega_j \zeta_j t\right) \frac{\sin \bar{\omega}_j t}{\bar{\omega}_j} \end{aligned}\right\} \quad (2\text{-}17)$$

thus $u(\underline{x}, t) = X_i(\underline{x}) y_i(t)$.

In the case of Subsection II-A-2,

$$\left.\begin{aligned} y_j(t) &= \int_0^t h_j(t-\tau)p(\tau)\, d\tau\ \delta_{ij} \\ h_j(t) &= \exp\,(-\omega_j\zeta_j t)\ \frac{\sin \bar{\omega}_j t}{\bar{\omega}_j} \end{aligned}\right\} \tag{2-18}$$

thus $\underline{u}(t) = \underline{\phi}^{(i)} y_i(t)$.

Hence we see that in both cases a pure normal mode is excited. In particular, if $p(t) = p_0 \cos \omega t$, then, as $t \to \infty$, we have

$$\left.\begin{aligned} u(\underline{x},\ t) &= \underline{X}_i(\underline{x}) \frac{p_0 \cos(\omega t - \alpha_i)}{\sqrt{\left(\omega_2^2 - \omega^2\right)^2 + (2\omega_i\omega\zeta_i)^2}} \\ &\text{or} \\ u(t) &= \underline{\phi}^{(i)} \frac{p_0 \cos(\omega t - \alpha_i)}{\sqrt{\left(\omega_i^2 - \omega^2\right)^2 + (2\omega_i\omega\zeta_i)^2}} \end{aligned}\right\} \tag{2-19}$$

where

$$\alpha_i = \tan^{-1} \frac{2\omega_i\omega\zeta_i}{\omega_i^2 - \omega^2}$$

Using the result of Equation (2-19), we can determine ω_i, ζ_i, and $\underline{\phi}^{(i)}$.

4. Nonclassical Normal Modes

For simplicity, we shall restrict the discussion to discrete, viscously damped systems. If in Subsection II-A-2 $M^{-1}D$ and $M^{-1}K$ do not commute, then classical normal modes do not exist and it will be shown that it is impossible to excite pure eigenmodes by any choice of real forcing functions. It should be noted that even in this case it is possible to excite "fairly pure" eigenmodes.

The formulation of 2N space is as follows:

Let $\underline{z} = \begin{pmatrix} \underline{x} \\ \dot{\underline{x}} \end{pmatrix}$, then Equation (2-8) can be rewritten in the form

$$\dot{\underline{z}} = A\underline{z} + \underline{b}(t)$$

$$\underline{z}(0) = \underline{0}$$

where

$$\left.\begin{aligned} A &= \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} \\ &\text{and} \\ \underline{b}(t) &= \begin{pmatrix} 0 \\ M^{-1}\underline{f}(t) \end{pmatrix} \end{aligned}\right\} \tag{2-20}$$

If the matrix A is nondefective, there exists a nonsingular matrix T such that:

$$T^{-1}AT = \Lambda = \begin{bmatrix} \Lambda_1 & \\ & \bar{\Lambda}_1 \end{bmatrix} \tag{2-21}$$

where

$$\Lambda_1 = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & & & \lambda_N \end{bmatrix} \tag{2-22}$$

is a diagonal matrix of complex eigenvalues. The matrix T has the structure:

$$T = \left[\underline{t}^{(1)},\ \underline{t}^{(2)},\ \cdots,\ \underline{t}^{(2N)}\right]$$

$$= \left[\begin{array}{c|c} \mathbf{\Phi} & \bar{\mathbf{\Phi}} \\ \hline \mathbf{\Phi}\Lambda_1 & \overline{\mathbf{\Phi}\Lambda_1} \end{array}\right]$$

where

$$\mathbf{\Phi} = \left[\underline{\phi}^{(1)},\ \underline{\phi}^{(2)},\ \cdots,\ \underline{\phi}^{(N)}\right]$$

and

$$\left(\lambda_i^2 M + \lambda_i D + K\right) \underline{\phi}^{(i)} = 0 \qquad i \ \varepsilon \left[1,\ N\right] \tag{2-23}$$

In general, the $\underline{\phi}^{(i)}$ are complex N vectors. The inverse of T is given by:

$$T^{-1} = \left[\begin{array}{c|c} C_1 & C_2 \\ \hline C_3 & C_4 \end{array}\right]$$

where

$$C_1 = \left(\mathbf{\Phi}\Lambda_1\right)^{-1}\left[\mathbf{\Phi}\Lambda_1^{-1}\phi^{-1} - \overline{\mathbf{\Phi}\Lambda_1^{-1}\mathbf{\Phi}^{-1}}\right]^{-1}$$

$$C_3 = \bar{C}_1$$

$$C_2 = \mathbf{\Phi}^{-1}\left[\mathbf{\Phi}\Lambda_1\mathbf{\Phi}^{-1} - \overline{\mathbf{\Phi}\Lambda_1\mathbf{\Phi}^{-1}}\right]^{-1}$$

$$C_4 = \bar{C}_2 \tag{2-24}$$

Let

$$\underline{z} = T\underline{y} \tag{2-25}$$

Then

$$\left.\begin{aligned} \dot{\underline{y}} &= \Lambda\underline{y} + \underline{q}(t) \\ \underline{y}(0) &= \underline{0} \\ \underline{q}(t) &= T^{-1}\underline{b}(t) \end{aligned}\right\} \tag{2-26}$$

$$\therefore \quad \underline{y}(t) = \int_0^t \begin{bmatrix} \exp\left(\Lambda_1(t - \tau)\right) & 0 \\ 0 & \exp\left(\bar{\Lambda}_1(t - \tau)\right) \end{bmatrix} T^{-1}\underline{b}(\tau)\, d\tau \tag{2-27}$$

Using Equations (2-24) and (2-25), we have

$$\underline{x}(t) = 2\mathrm{Re} \int_0^t \Phi \exp\left(\Lambda_1(t - \tau)\right)\Phi^{-1}\left[\Phi\Lambda_1\Phi^{-1} - \overline{\Phi\Lambda_1\Phi^{-1}}\right]^{-1} M^{-1}\underline{f}(\tau)\, d\tau \tag{2-28}$$

$$\therefore \quad \underline{x}(t) = \int_0^t \mathrm{Im}\left(\Phi \exp\left(\Lambda_1(t - \tau)\right)\Phi^{-1}\right)\left[\mathrm{Im}\left(\Phi\Lambda_1\Phi^{-1}\right)\right]^{-1} M^{-1}\underline{f}(\tau)\, d\tau \tag{2-29}$$

Since $M^{-1}\underline{f}(t)$ is a real vector, and $\mathrm{Im}[\Phi \exp(\Lambda_1 t)\Phi^{-1}]$ and $\mathrm{Im}(\Phi\Lambda_1\Phi^{-1})$ are real matrices, no choice of the forcing function $\underline{f}(t)$ will result in the excitation of a <u>pure</u> eigenmode. It may also be shown that it is impossible to excite a pair of complex conjugate <u>pure</u> eigenmodes. Thus, unlike the system of Subsections II-A-1 and II-A-2, modal testing does not enable us to accurately identify the eigenvalues and eigenvectors of systems with non-classical normal modes. Despite this fact, if in Equation (2-8) the damping matrix D is small and the eigenvalues well separated, Equation (2-29) can be evaluated approximately.

Let Φ_0 be such that

$$\Phi_0^T M \Phi_0 = I; \quad \Phi_0^T K \Phi_0 = \left[\diagdown \omega^2 \diagdown \right]; \quad \Phi_0^T D \Phi_0 = \mathscr{D} = \mathscr{D}^T \tag{2-30}$$

If we set

$$\underline{f}(t) = M \underline{\phi}_0^{(i)} p_0 \cos \omega t \tag{2-31}$$

and in Equation (2-30)

$$|\mathscr{D}_{ij}| << \omega_i^2 \qquad \forall i, j \tag{2-32}$$

then after the initial transients die out,

$$\underline{x}(t) \simeq \frac{p_0}{\sqrt{\left(\omega_i^2 - \omega^2\right)^2 + \left(\mathscr{D}_{ii}\omega\right)^2}} \left\{ \underline{\phi}_0^{(i)} \cos(\omega t - \alpha_i) + \sum_{k=1}^{N}{}' \frac{\omega \mathscr{D}_{ki} \sin(\omega t - \alpha_i - \alpha_k)}{\sqrt{\left(\omega_k^2 - \omega^2\right)^2 + (\mathscr{D}_{kk}\omega)^2}} \underline{\phi}_0^{(k)} \right\} \tag{2-33}$$

$$\alpha_j = \tan^{-1} \frac{\omega \mathscr{D}_{jj}}{\omega_j^2 - \omega^2}$$

If

$$|\mathscr{D}_{ij}| << \omega_1^2 \qquad \forall i, j$$

and ω_i and ω_j are distinct and well separated, and $\omega \cong \omega_i$, then from Equation (2-33) the main effect of nonclassical damping is to cause phase shifts in the response vector. Each mass no longer passes through its equilibrium portion at the same time as all the other masses, as was the case for classically damped systems. In particular, Equation (2-33) shows that if

$$\frac{\omega\omega_k}{|\omega_k^2 - \omega^2|} \frac{|\mathscr{D}_{ki}|}{\omega_k} << 1$$

then

$$\underline{x}(t) \simeq \frac{p_0 \cos(\omega t - \alpha_i)}{\sqrt{\left(\omega_i^2 - \omega^2\right)^2 + \left(\omega \mathscr{D}_{ii}\right)^2}} \underline{\phi}_0^{(i)} \qquad (2\text{-}34)$$

Thus, for small damping, the response is almost a pure normal mode. As the damping increases and the separation between the eigenvalues decreases, the effects of nonclassical damping become stronger and the response is no longer, even approximately, in a pure normal mode (Reference 2-2).

5. Useful Properties of Discrete Systems with Classical Normal Modes

Returning to System (2-8) in the case of Subsection II-A-2, the Properties (2-9), (2-10), and (2-11) can be rewritten in the form

$$\left.\begin{aligned} \mathbf{\Phi} &= \left[\underline{\phi}^{(1)}, \underline{\phi}^{(2)}, \cdots \underline{\phi}^{(N)}\right] \\ \mathbf{\Phi}^T M \mathbf{\Phi} &= I \\ \mathbf{\Phi}^T D \mathbf{\Phi} &= \left[\diagdown\, 2\omega_i\zeta_i \diagdown\right] \\ \mathbf{\Phi}^T K \mathbf{\Phi} &= \left[\diagdown\, \omega_i^2 \diagdown\right] \end{aligned}\right\} \qquad (2\text{-}35)$$

Since the vectors $\phi^{(i)}$ are linearly independent, the matrix ϕ is nonsingular; therefore:

$$M = (\boldsymbol{\Phi}^{-1})^T (\boldsymbol{\Phi}^{-1})$$

$$D = M\boldsymbol{\Phi} \left[\diagdown 2\omega_i\zeta_i \diagdown \right] \boldsymbol{\Phi}^T M \qquad (2\text{-}36)$$

$$K = M\boldsymbol{\Phi} \left[\diagdown \omega_i^2 \diagdown \right] \boldsymbol{\Phi}^T M$$

B. SYSTEM REDUCTION

We shall now prove two interesting theorems.

1. Theorem I

A continuous dynamical system, such as that of Subsection II-A-1, is given. It exhibits a complete set of linearly independent viscously damped classical normal modes having eigenfunctions $X_i(\underline{x})$, and eigenvalues ω_i, $i \in [1, \infty]$. Given a positive integer N, there exists an N^{th}-order viscously damped discrete system exhibiting a complete set of classical normal modes having the property that its i^{th} eigenvalue $\tilde{\underline{\omega}}_i$ corresponds exactly to the i^{th} eigenvalue $\underline{\omega}_i$ of the continuous system and further the i^{th} eigenvector $\underline{\phi}^{(i)}$ corresponds to a projection of the i^{th} eigenfunction of the continuous system. That is,

$$\tilde{\omega}_i = \omega_i \qquad \phi_j^{(i)} = X_i(\underline{x}_j)$$

$$i, j \in [1, N] \qquad \underline{x}_j \in \mathscr{D} \qquad (2\text{-}37)$$

Proof: Since the eigenfunctions of the continuous problem are linearly independent, the function

$$g(\underline{x}) = \sum_{i=1}^{N} \alpha_i X_i(\underline{x}) \qquad (2\text{-}38)$$

cannot vanish identically unless $\alpha_i \equiv 0 \quad \forall i \; \varepsilon \; (1, N)$. Hence, there exists N points $\underline{x}_j$ $j \; \varepsilon \; (1, N)$ such that the vectors

$$\underline{\phi}^{(i)} = \left\{ X_i(\underline{x}_j) \right\} \qquad i, j \; \varepsilon \; (1, N) \tag{2-39}$$

are linearly independent.

Let

$$\mathbf{\Phi} = \left[\underline{\phi}^{(1)}, \underline{\phi}^{(2)}, \cdots \underline{\phi}^{(N)} \right] \tag{2-40}$$

Since the $\underline{\phi}^{(i)}$ are linearly independent, the matrix $\underline{\mathbf{\Phi}}$ is nonsingular. Let

$$\left. \begin{aligned} M &= \alpha^2 (\mathbf{\Phi}^{-1})^T (\mathbf{\Phi}^{-1}) \\ D &= \frac{1}{\alpha^2} M\mathbf{\Phi} \left[\diagdown \; 2\omega_i \zeta_i \; \diagdown \right] \phi^T M \\ K &= \frac{1}{\alpha^2} M\mathbf{\Phi} \left[\diagdown \; \omega_i^2 \; \diagdown \right] \phi^T M \end{aligned} \right\} \tag{2-42}$$

where α^2 is chosen such that

$$T_r M = \int_{\mathscr{D}} \rho(x) \; dx \tag{2-42}$$

Using M, D, and K constructed as in Equation (2-41) to form the N^{th}-order discrete system,

$$\begin{aligned} M\ddot{\underline{u}} + D\dot{\underline{u}} + K\underline{u} &= \underline{\tilde{f}}(t) = M\mathbf{\Phi}\underline{\tilde{q}}\,(t) \\ \underline{u}(0) &= \dot{\underline{u}}(0) = 0 \end{aligned} \tag{2-43}$$

This system has the following properties:

(1) There exists a complete set of ordinary eigenvectors $\underline{\phi}^{(i)}$ $i \varepsilon (1, N)$ such that if $\mathbf{\Phi} = \left[\underline{\phi}^{(1)}, \underline{\phi}^{(2)} \cdots \underline{\phi}^{(N)}\right]$ as in Equation (2-40), then

$$\mathbf{\Phi}^T M \mathbf{\Phi} = \alpha^2 I$$

(2) $\mathbf{\Phi}^T D \mathbf{\Phi} = \left[\diagdown\, 2\omega_i\zeta_i \,\diagdown\right]\alpha^2$

(3) $\mathbf{\Phi}^T K \mathbf{\Phi} = \left[\diagdown\, \omega_i^2 \,\diagdown\right]\alpha^2$

(4) If $f(\underline{x}, t) = \sum_{i=1}^{N} \tilde{q}_i(t) X_i(\underline{x}) \rho(\underline{x})$

then $\tilde{q}_i(t) = \int_{\mathscr{D}} \rho(\underline{x}) f(\underline{x}, t)\, X_i(\underline{x})\, d\underline{x}$ (2-44)

If in Equation (2-43) $\underline{u} = \mathbf{\Phi}\underline{y}$, Equation (2-43) is reduced to:

$$\ddot{y}_i + 2\omega_i\zeta_i\dot{y}_i + \omega_i^2 y_i = \tilde{q}_i(t) \qquad (2\text{-}45)$$

which is exactly the same as the i^{th} mode of the continuous system of Sub-section II-A-1 with $f(\underline{x}, t)$ given by Equation (2-44).

Hence, since

$$\underline{u}(t) = \mathbf{\Phi}\underline{y}(t) \qquad (2\text{-}46)$$

then

$$u_j(t) = \sum_{i=1}^{N} \phi_j^{(i)} y_i(t) \tag{2-47}$$

$$= \sum_{i=1}^{N} X_i(\underline{x}_j) y_i(t) \tag{2-48}$$

$$\therefore \quad u_j(t) \equiv u(\underline{x}_j, t) \tag{2-49}$$

That is, the solution of the discrete Problem (2-38) is the projection of the solution of the continuous problem of Subsection II-A-1 with $f(\underline{x},t)$ given by Equation (2-44).

It should be noted that there exist, in general, infinite sets of points $\{\underline{x}_j\}$ $j \in [1, N]$ that may be used to define the sets of vectors $\{\underline{\phi}^{(i)}\}$ $i \in [1, N]$. Thus there exist infinitely many N^{th}-order models that can be used to mimic the behavior of the continuous system. It is not surprising, therefore, that observations at N points in a continuous system do not permit unique identification of the continuous system.

2. Theorem II

Given a discrete N^{th}-order dynamical system exhibiting a complete set of linearly independent viscously damped classical normal modes $\underline{\phi}^{(i)}$, $i \in [1, N]$. Given any positive integer $N_2 < N$, there exists an N_2^{th}-order discrete dynamical system exhibiting a complete set of linearly independent classical normal modes $\underline{\psi}^{(j)}$, $j \in [1, N_2]$, having the property that its i^{th} eigenvalue corresponds exactly to the i^{th} eigenvalue of the larger system and further, the i^{th} eigenvector $\underline{\psi}^{(i)}$ corresponds exactly to a projection of the i^{th} eigenvector $\underline{\phi}^{(i)}$ of the larger system.

Proof: Since the eigenvectors of the given system are linearly independent, the matrix

$$\mathbf{\Phi} = \left[\underline{\phi}^{(1)}, \underline{\phi}^{(2)}, \cdots, \underline{\phi}^{(N)} \right] \tag{2-50}$$

is nonsingular, that is,

$$|\mathbf{\Phi}| \neq 0 \tag{2-51}$$

However, if this is true, there must exist nonvanishing minors of all orders less than N. In particular, there must exist at least one nonvanishing minor of order N_2. Let

$$\underline{\psi}^{(i)} = S\, \underline{\phi}^{(i)} \quad i \in \left[1, N_2\right] \tag{2-52}$$

Where S is an $N_2 \times N$ matrix whose columns consist of either the null vector $\underline{0}$, or distinct unit vectors $\underline{e}_j$, $j \in \left[1, N_2\right]$ have zero entries in all but the j^{th} row, which has unity. The matrix

$$\mathbf{\Psi} = \left[\underline{\psi}^{(1)}, \underline{\psi}^{(2)} \cdots, \underline{\psi}^{(N_2)} \right] \tag{2-53}$$

is such that $|\mathbf{\Psi}|$ is a minor of $\mathbf{\Phi}$ of order N_2 and so does not vanish for an appropriate choice of the matrix S. Let

$$\begin{aligned} M_2 &= \alpha^2 (\mathbf{\Psi}^{-1})^T (\mathbf{\Psi}^{-1}) \\ D_2 &= \frac{1}{\alpha^2} M_2 \mathbf{\Psi} \left[\diagdown 2\omega_j \zeta_j \diagdown \right] \mathbf{\Psi}^T M_2 \\ K_2 &= \frac{1}{\alpha^2} M_2 \mathbf{\Psi} \left[\diagdown \omega_j^2 \diagdown \right] \mathbf{\Psi}^T M_2 \end{aligned} \tag{2-54}$$

where α^2 is chosen so that

$$T_r M_2 = T_r M \tag{2-55}$$

Using the M_2, D_2, and K_2 constructed in Equation (2-54) to form the N_2^{th}-order discrete system,

$$\left.\begin{aligned} M_2\ddot{\underline{u}}_2 + D_2\dot{\underline{u}}_2 + K_2\underline{u}_2 &= f_2(t) \\ \underline{u}_2(0) = \dot{\underline{u}}_2(0) &= 0 \end{aligned}\right\} \qquad (2\text{-}56)$$

This system has the following properties:

(1) There exists a complete set of ordinary eigenvectors $\underline{\psi}^{(j)}\ j \ \varepsilon \ [1, N_2]$ such that if $\boldsymbol{\Psi} = \left[\underline{\psi}^{(1)}, \underline{\psi}^{(2)} \cdots \underline{\psi}^{(N_2)}\right]$, then $\boldsymbol{\Psi}^T M_2 \boldsymbol{\Psi} = \alpha^2 I$

(2) $\boldsymbol{\Psi}^T D_2 \boldsymbol{\Psi} = \begin{bmatrix} \ddots & & \\ & 2\omega_j\zeta_j & \\ & & \ddots \end{bmatrix} \alpha^2$

(3) $\boldsymbol{\Psi}^T K_2 \boldsymbol{\Psi} = \begin{bmatrix} \ddots & & \\ & \omega_j^2 & \\ & & \ddots \end{bmatrix} \alpha^2$

(4) If $\underline{f}(t) = \sum_{i=1}^{N_2} M\underline{\phi}^{(i)}\tilde{q}_i(t)$, then

$$\underline{f}_2(t) = \sum_{i=1}^{N_2} M_2\underline{\psi}^{(i)}\tilde{q}_i(t) \qquad (2\text{-}57)$$

If $\underline{u}_2 = \boldsymbol{\Psi}\underline{z}$, then Equation (2-56) is reduced to: (2-58)

$$\left.\begin{aligned} \ddot{z}_i + 2\omega_i\zeta_i\dot{z}_i + \omega_i^2 z_i &= \tilde{q}_i(t) \\ z_i(0) = \dot{z}_i(0) &= 0 \end{aligned}\right\} \qquad (2\text{-}59)$$

Equation (2-59) is exactly the same as the i^{th} mode of the large system. Hence, since

$$\underline{u}_2(t) = \Psi \underline{z}(t) \tag{2-60}$$

$$u_{2i}(t) = \sum_{i=1}^{N_2} \psi_i^{(j)} z_j(t) \tag{2-61}$$

Using Equation (2-52), we have

$$\left.\begin{aligned} u_{2i}(t) &= S \sum_{j=1}^{N_2} \underline{\phi}_i^{(j)} z_j(t) \\ &= S\underline{x}_i(t) \end{aligned}\right\} \tag{2-62}$$

$$\therefore \quad \underline{u}_2(t) = S\underline{x}(t) \tag{2-63}$$

That is, the solution of the lower-order system is the projection of the solution of the higher-order system.

It should be noted that there exist, in general for each $N_2 < N$, several nonvanishing minors. Therefore, there exist several N_2^{th}-order models that mimic the behavior of the N^{th}-order system. It is not surprising, therefore, that observations at N_2 points in a N^{th}-order discrete system do not, in general, permit unique identification of the N^{th}-order system, unless $N_2 \equiv N$ or there exists some special structure to the system.

Theorems I and II ensure that there exist finite-dimension lumped parameter models that can mimic exactly the behavior of a higher-order lumped parameter or continuous system under appropriate conditions.

In practice we observe the response of a system at only a small finite number of points; the response at these points is clearly a projection of the response of the total system. Furthermore, we can, in general, apply forces to the structure at only a small finite number of points. By limiting the number of points at which we excite the structure and observe the response, we prejudice the outcome of any attempt to identify the structure uniquely. In general, if we excite a structure and observe its response at N_2 points, we can uniquely identify only an N_2^{th}-order lumped parameter model.

C. EFFECT OF MODELING ERRORS ON PREDICTED RESPONSE

Given the System (2-64),

$$\left.\begin{aligned} &M\ddot{\underline{x}} + D\dot{\underline{x}} + K\underline{x} = \begin{cases} \underline{f}(t) & t \le T \\ \underline{0} & t > T \end{cases} \\ &\underline{x}(0) = \dot{\underline{x}}(0) = \underline{0} \end{aligned}\right\} \tag{2-64}$$

It is sometimes more convenient to write the equations in 1st-order form.

$$\left.\begin{aligned} &\frac{d\underline{z}}{dt} = A\underline{z} + \begin{cases} \underline{g}(t) & t \le T \\ \underline{0} & t > T \end{cases} \\ &\underline{z}(0) = \underline{0} \\ &\text{where} \\ &A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} \\ &\text{and} \\ &\underline{g}(t) = \begin{Bmatrix} 0 \\ M^{-1}\underline{f}(t) \end{Bmatrix} \end{aligned}\right\} \tag{2-65}$$

Suppose now that we have an analytical model of System (2-65).

$$\left.\begin{aligned} &\frac{d\underline{y}}{dt} = B\underline{y} + \begin{cases} \underline{h}(t) & t \le T \\ 0 & t > T \end{cases} \\ &\underline{y}(0) = \underline{0} \end{aligned}\right\} \qquad (2\text{-}66)$$

where

$$B \approx A \qquad \underline{h}(t) \approx \underline{g}(t) \qquad (2\text{-}67)$$

We wish to know what errors are induced in the solution by modeling errors in A and $\underline{g}(t)$. Let

$$\underline{w} = \underline{z} - \underline{y} \qquad (2\text{-}68)$$

thus

$$\frac{d\underline{w}}{dt} = A\underline{w} + (B - A)\underline{y} + \left\{\underline{g}(t) - \underline{h}(t)\right\} \qquad (2\text{-}69)$$

$$\underline{w}(0) = \underline{0} \qquad (2\text{-}70)$$

thus

$$\underline{w}(t) = \int_0^{\mathrm{Min}(t,T)} \exp\ A(t - \tau)\ \left[(B - A)\underline{y}(\tau) + \left(\underline{g}(\tau) - \underline{h}(\tau)\right)\right] d\tau \quad (2\text{-}70)$$

thus

$$||\underline{w}(t)|| \le \int_0^{\mathrm{Min}(t,T)} ||\exp\ A(t - \tau)|| \Big[||(B - A)||\ ||\underline{y}(\tau)||$$

$$+ ||\ \underline{g}(\tau) - h(\tau)||\Big]\ d\tau \qquad (2\text{-}71)$$

Now

$$\left.\begin{aligned} ||\exp\ (At)|| &\le M_1\ \exp(-\alpha_1 t) \\ ||\exp\ (Bt)|| &\le M_2\ \exp(-\alpha_2 t) \qquad \alpha_1,\ \alpha_2 > 0 \end{aligned}\right\} \tag{2-72}$$

Let

$$\left.\begin{aligned} M &= \text{Max}\ (M_1,\ M_2) \\ \alpha &= \text{Min}\ (\alpha_1,\ \alpha_2) \\ d &= \sup_t\ ||\underline{g}(t) - \underline{h}(t)|| \\ k &= \text{Max}\left[\sup_t ||\underline{g}(t)||,\ \sup_t\ ||\underline{h}(t)||\right] \end{aligned}\right\} \tag{2-73}$$

Then

$$||w(t)|| \le \left[||B - A||\ \sup_t\ ||\underline{y}(t)|| + d\right] \frac{M}{\alpha} \left[\exp\ \left(-\alpha(t - t^*)\right) - \exp\ (-\alpha t)\right] \tag{2-74}$$

where

$$\left.\begin{aligned} t^* &= t \text{ if } t < T \\ &= T \text{ if } t \ge T \end{aligned}\right\} \tag{2-75}$$

Now

$$\underline{y}(t) = \int_0^{\text{Min}(t,T)} \exp\ \left(B(t - \tau)\ \underline{h}(\tau)\right)\ d\tau \tag{2-76}$$

$$\therefore \quad ||\underline{y}(t)|| \leq \frac{M}{\alpha} \sup_t ||h(\tau)|| \left[\exp\left(-\alpha(t - t^*)\right) - \exp(-\alpha t)\right] \tag{2-77}$$

$$\therefore \quad ||\underline{w}(t)|| \leq \left(\frac{M}{\alpha}\right)^2 k \left(\exp\left(-\alpha(t - t^*)\right) - \exp(-\alpha t)\right)^2 ||B - A|| \tag{2-78}$$

$$+ \frac{M}{\alpha} d \left(\exp\left(-\alpha(t - t^*)\right) - \exp(-\alpha t)\right)$$

There are two cases of special interest.

1. Case (a) - Short Transient Loads

   If $\alpha T << 1$, then Equation (2-78) yields:

$$\sup_t ||\underline{w}(t)|| \leq (MT)^2 k ||B - A|| + MTd \tag{2-79}$$

If $MT \sim 0(1)$, then the two terms d and $k\ ||B - A||$ are of equal importance. Hence, the errors $||B - A||$ and $||\underline{g}(t) - \underline{h}(t)||$ are of equal significance and the system parameters need not be known with any higher accuracy than the forcing functions.

2. Case (b) - Persistent Excitation

   If $\alpha T >> 1$, then Equation (2-78) yields:

$$\sup_t ||\underline{w}(t)|| \leq \left(\frac{M}{\alpha}\right)^2 k\ ||B - A|| + \left(\frac{M}{\alpha}\right) d \tag{2-80}$$

Since in many structural dynamics problems the damping is small, the term M/α becomes very large compared to unity. In this case, the first term $k||B - A||$ assumes much greater importance than the second term d; this shows the possible effects of resonance. Thus we see that under persistent excitation, system errors can play a dominant roll; hence, the system parameters must be defined with a much higher accuracy than the forcing functions. It is interesting to note that Chen and Wada (Reference 2-3) established a similar result using perturbational analyses.

D. EFFECTS OF NEGLECTING HIGHER MODES

Let us first consider the case of an N^{th}-order discrete System (2-8) in which the forces $\underline{f}(t)$ are basically low frequency in nature

$$\left.\begin{aligned} M\ddot{\underline{u}} + D\dot{\underline{u}} + K\underline{u} &= f(t) \\ \underline{u}(0) = \dot{\underline{u}}(0) &= \underline{0} \end{aligned}\right\} \tag{2-81}$$

where $M^{-1}D$ and $M^{-1}K$ commute. Then

$$u(t) = \left\{ \Phi \left[\diagdown \int_0^t h_i(t-\tau) \diagdown \right] \Phi^T f(t)\, d\tau \right\} \tag{2-82}$$

$$u(t) = \sum_{i=1}^{N} \underline{\phi}^{(i)} z_i(t) \tag{2-83}$$

$$z_i(t) = \int_0^t \frac{\exp(-\omega_i \zeta_i (t-\tau))}{\bar{\omega}_i} \sin \omega_i (t-\tau)\, q_i(\tau)\, d\tau \tag{2-84}$$

where

$$\left.\begin{aligned} & q(t) = \Phi^T \underline{f}(t) \\ & \Phi^T K \Phi = \left[\diagdown \omega_i^2 \diagdown \right], \quad \Phi^T D \Phi = \left[\diagdown 2\omega_i \zeta_i \diagdown \right] \\ & \bar{\omega}_i = \omega_i \sqrt{1 - \zeta_i} \qquad i \in (1, N) \end{aligned}\right\} \tag{2-85}$$

If the frequency spectrum of $\underline{f}(t)$ contains only frequencies Ω_f which are small compared to all but the lowest eigenvalue ω_i, then

$$\Omega_f < \omega_i \quad \text{for} \quad i > N^*,\ N^* < N \tag{2-86}$$

In this case it is reasonable to approximate the solution $\underline{u}(t)$ by the truncated series,

$$\underline{u}(t) \simeq u_T(t) = \sum_{i=1}^{N^*} \underline{\phi}^{(i)} z_i(t) \tag{2-87}$$

If we look at a typical term in the remainder of the series,

$$\left.\begin{aligned} z_j(t) &= \int_0^t h_j(t - \tau)\, q_j(\tau)\, d\tau \\ &= \int_0^t h_j(\tau)\, q_j(t - \tau)\, d\tau \end{aligned}\right\} \tag{2-88}$$

Expanding $q_j(t - \tau)$ in a Taylor's series about $\tau = 0$, we find

$$z_j(t) = q_j(t) \int_0^t h_j(\tau)\, d\tau - \dot{q}_j(t) \int_0^t \tau h_j(\tau)\, d\tau$$

$$+ \frac{\ddot{q}_j(t)}{2} \int_0^t \tau^2 h_j(\tau)\, d\tau + \cdots \tag{2-89}$$

If $\omega_j \zeta_j t > 1$, then

$$\left.\begin{aligned} \int_0^t h_j(\tau)\, d\tau &\simeq \frac{1}{\omega_j^2} \\ \int_0^t \tau h_j(\tau)\, d\tau &\simeq \frac{2\zeta_j}{\omega_j^3} \\ \int_0^t \tau^2 h_j(\tau)\, d\tau &\simeq \frac{2 - 4\zeta_j^2}{\omega_j^4} \end{aligned}\right\} \tag{2-90}$$

Thus

$$\left| z_j(t) - \frac{q_j(t)}{\omega_j^2} \right| \leq \frac{\left|\dot{q}_j(t)\right|_{max} 2\zeta_i}{\omega_j^3} \tag{2-91}$$

If

$$\frac{\left|\dot{q}_j(t)\right|_{max}}{\omega_j \left|q_j(t)\right|_{max}} < 1 \tag{2-92}$$

i.e., $q_j(t)$ is low frequency compared to ω_j, then

$$z_j(t) \simeq \frac{q_j(t)}{\omega_j^2}, \quad j \in (N^* + 1, N) \tag{2-93}$$

In this case we may improve the approximate solution of Equation (2-87) by adding the additional terms of Equation (2-93).

$$\therefore \quad \underline{u}(t) \simeq \sum_{j=1}^{N^*} \underline{\phi}^{(j)} z_j(t) + \sum_{N^*+1}^{N} \underline{\phi}^{(j)} z_j(t) \tag{2-94}$$

Let

$$z_i^{(s)}(t) = \frac{q_i(t)}{\omega_i^2} \quad i \in (1, N) \tag{2-95}$$

Equation (2-94) may be rewritten,

$$u(t) \simeq \sum_{j=1}^{N^*} \underline{\phi}^{(j)} \left[z_j(t) - z_j^{(s)}(t) \right] + \sum_{j=1}^{N} \underline{\phi}^{(j)} z^{(s)}(t) \tag{2-96}$$

Now

$$\underline{q}^{(t)} = \Phi^T \underline{f}(t) \tag{2-97}$$

$$\therefore \quad \sum_{j=1}^{N} \underline{\phi}^{(j)} z_j^{(s)}(t) = \Phi \begin{bmatrix} \diagdown \frac{1}{\omega_i^2} \diagdown \end{bmatrix} \Phi^T \underline{f}^{(t)} \tag{2-98}$$

But

$$\Phi^T K \Phi = \begin{bmatrix} \diagdown \omega_i^2 \diagdown \end{bmatrix} \tag{2-99}$$

$$\therefore \quad K^{-1} = \Phi \begin{bmatrix} \diagdown 1/\omega_i^2 \diagdown \end{bmatrix} \Phi^T \tag{2-100}$$

$$\therefore \quad \sum_{i=1}^{N} \underline{\phi}_i z_i^{(s)}(t) = K^{-1} \underline{f}(t) = \underline{u}^{(s)}(t) \tag{2-101}$$

where $\underline{u}^{(s)}(t)$ is the "static" response of the system to the applied forces $\underline{f}(t)$.

If we write

$$\underline{u}^{(d)}(t) = \sum_{i=1}^{N^*} \underline{\phi}^i \left[z_i(t) - z_i^{(s)}(t) \right] \tag{2-102}$$

then

$$\underline{u}(t) = \underline{u}^{(d)}(t) + \underline{u}^{(s)}(t) \tag{2-103}$$

Thus we see that the total response is approximated by two separate terms; $\underline{u}^{(d)}(t)$ consists of the dynamic response of the active modes less the "static" response in these modes and $u^{(s)}(t)$ the "static" response of the whole structure to the applied forces $\underline{f}(t)$. These results are identical to what is sometimes called the mode acceleration method.

As a measure of the error in neglecting the higher modes we have

$$\eta = \frac{||\underline{u}(t)|| - ||\underline{u}_T(t)||}{||\underline{u}(t)||} \tag{2-104}$$

This error tends to be smaller for the case of persistent excitation where resonance may occur, and higher for short transients where resonance has no chance of occurring.

E. ERRORS IN EIGENVALUES AND EIGENVECTORS

From an analytical point of view, errors in eigenvalues and eigenvectors are usually the result of modeling errors or the use of too crude a level of discretization of the continuous system. In practice, it is usually possible to model the lower eigenvalues and eigenvectors of aerospace structures with an accuracy of five to ten percent, or better. The eigenvalues are usually more accurately modeled than the eigenvectors though this may be simply a problem related to the difficulties encountered in testing and measuring eigenvalues and eigenvectors. This subject will be treated in more detail in Section III of this report.

F. ERRORS IN MODAL "FORCE" COEFFICIENTS

Errors in eigenvalues and eigenvectors have a significant effect on the modal "force" coefficients; however, even if the eigenvalues and eigenvectors of the first N modes are known exactly, errors in the modal force coefficients will still arise due to discretization effects. For example, a central difference approximation may be used to calculate the curvature of a beam using the discrete displacements of the beam.

Thus $\partial^2 u/\partial x^2$ is approximated by

$$\frac{u_{n+1} + u_{n-1} - 2u_n}{h} \tag{2-105}$$

where $u_n = u(x_n)$, $x_n = nh$, and h is the mesh spacing. Now

$$\frac{u_{n+1} + u_{n-1} - 2u_n}{h^2} = \frac{\partial^2 u}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4} + 0(h^4) \tag{2-106}$$

Thus

$$\left|\frac{u_{n+1} + u_{n-1} - 2u_n - h^2 \dfrac{\partial^4 u}{\partial x^2}}{h^2 \left|\dfrac{\partial^2 u}{\partial x^2}\right|_{max}}\right|_{max} \leq \frac{h^2}{12}\,\frac{\left|\dfrac{\partial^4 u}{\partial x^4}\right|_{max}}{\left|\dfrac{\partial^2 u}{\partial x^2}\right|_{max}} \tag{2-107}$$

If

$$u(x) = A \sin\left[\frac{2\pi x}{\lambda}\right] \tag{2-108}$$

then

$$\left|\frac{u_{n+1} + u_{n-1} - 2u_n - h^2 \dfrac{\partial^2 u}{\partial x^2}}{h^2 \left|\dfrac{\partial^2 u}{\partial x^2}\right|_{max}}\right|_{max} \leq \frac{\pi^2}{3}\left(\frac{h}{\lambda}\right)^2 \tag{2-109}$$

Now $\lambda/h = N_m$, the number of mesh points per wavelength;

$$\therefore \quad \left| \frac{u_{n+1} + u_{n-1} - 2u_n - h^2 \dfrac{\partial^2 u}{\partial x^2}}{h^2 \left| \dfrac{\partial^2 u}{\partial x^2} \right|_{max}} \right|_{max} \leq \frac{\pi^2}{3N_m^2} \qquad (2\text{-}110)$$

since

$$\sigma_B = E\, z \frac{\partial^2 u}{\partial x^2} \qquad (2\text{-}111)$$

The relative error in the bending stress is also given by Equation (2-110). Thus, the relative error in the bending stress increases rapidly as N_m, the number of mesh points/wavelength, is decreased. Since we have shown in Section II-B that it is theoretically possible to construct discrete models whose eigenvalues agree exactly with the first N eigenvalues of the continuous system, and that eigenvectors are projections of the eigenfunctions of the continuous system, it should not be too surprising that the accuracy of modal "force" coefficients is usually lower than that of either the eigenvalues or the eigenvectors.

In addition to the discretization errors discussed above, additional errors arise because of modal spill-over caused by experimental difficulties in obtaining pure modal excitation, and the fact that the real structure may not admit classical normal modes; these errors will be discussed in more detail in Section III of this report.

SECTION III
DYNAMIC TESTING

Though it is possible, using modern analytical and numerical techniques, to model the dynamic behavior of a structure to any desired degree of accuracy, few engineers would be happy to commission a new space vehicle without at least some limited dynamic testing. The main reasons for this conservative approach are: (1) while it is possible in theory to model the structure accurately, it is usually too costly or too time consuming to do so; (2) it is very easy to omit some significant effects such as geometric or material nonlinearities in modeling the system. For these reasons, most engineers require at least a limited program of dynamic testing to "qualify" the analytical model. If the limited program of testing yields good agreement between measured and predicted values, the engineer is happy; unfortunately, if the agreement is poor, as it frequently is, the engineer is left in a quandry. It has been suggested that the test data be used to update the analytical model and so increase the precision of the analytical predictions. This is a very useful technique and can yield good results if properly applied. First of all, it must be pointed out that due to the nonuniqueness of the identification process, it cannot be used to identify the parameters of the complete structural system. It can, however, be used to obtain updated estimates of the natural frequencies, damping factors, and mode shapes of the finite numbers of modes observed, and hence to make better estimates of the contributions of these modes to the response of the structure. While this approach is useful in improving the analytical and predictive capabilities for a given physical structure, it cannot help improve the analytical and predictive capabilities for new and unbuilt structures.

A. MODAL TESTING

As shown in Subsection II-A-3, discrete and continuous systems exhibiting classical normal modes are capable of being excited in pure normal modes. As shown in Subsection II-A-4, discrete systems exhibiting nonclassical normal modes cannot be excited in pure normal modes. Despite this fact, O'Kelly (Reference 2-2) has shown that if the damping in a structure is small and the eigenvalues well separated, the lower modes of any viscously damped structure can be excited in rather good approximations to pure modes.

In structural dynamics, it is commonly assumed that the system possesses classical normal modes. While this is seldom strictly true, the damping in many aerospace structures is often quite small, and so fairly pure modes can be excited, particularly in the lower modes.

Consider the N^{th}-order discrete system that exhibits classical normal modes:

$$M\ddot{\underline{u}} + D\dot{\underline{u}} + K\underline{u} = \underline{f}(t)$$
$$\underline{u}(t_0) = \dot{\underline{u}}(t_0) = \underline{0} \tag{3-1}$$

Let

$$\underline{f}(t) = \underline{C} \cos \omega t, \qquad t_0 \to -\infty \tag{3-2}$$

Since Equation (3-1) has classical normal modes, the solution of Equation (3-1) with Equation (3-2) can be written

$$\underline{u}(t) = \mathrm{Re}\left(\Phi R \Phi^T \underline{C} \exp\,(j\omega t)\right) \tag{3-3}$$

Where Φ is the modal matrix (2-12) and R is the response matrix:

$$R = \left[\diagdown \quad \frac{\exp\,(-j\alpha_i)}{\sqrt{\left(\omega_i^2 - \omega^2\right)^2 + (2\omega_i\zeta_i\omega)^2}} \quad \diagdown\right] \tag{3-4}$$

Where α is defined in Equation (2-19).

Let

$$\Phi^T \underline{C} = \underline{q} \tag{3-5}$$

Then

$$\underline{u}(t) = \text{Re}\left(\boldsymbol{\Phi}R\underline{q}\ \exp\ (j\omega t)\right) \tag{3-6}$$

If

$$\underline{q} = q_k \underline{e}_k \tag{3-7}$$

then

$$\underline{u}(t) = \underline{\phi}^{(k)} q_k \cos\ (\omega t - \alpha k) \Big/ \sqrt{\left(\omega_k^2 - \omega^2\right)^2 + (2\omega_k \zeta_k \omega)^2} \tag{3-8}$$

Condition (3-7) requires that

$$\underline{C} = q_k M \boldsymbol{\Phi} \underline{e}_k \tag{3-9}$$

However, $\boldsymbol{\Phi}$ is unknown at the beginning of the test.

We observe that if $\omega \cong \omega_k$, ζ_i << $\forall i$ and the eigenvalues ω_i are distinct and well separated, then $|R^{(k)}|$ >> $|R^{(i)}|$ $i \neq k$; thus:

$$\underline{u}(t) \simeq \underline{\phi}^{(k)} q_k \frac{\cos\ (\omega t - \alpha_k)}{\sqrt{\left(\omega_k^2 - \omega^2\right)^2 + (2\omega_k \zeta_k \omega)^2}} \tag{3-10}$$

This suggests that an iterative scheme can be used to generate pure modes. Using Equation (2-6), we can define an iterative scheme.

$$\left.\begin{aligned} \underline{u}_k(t) &= \text{Re}\ \underline{\tilde{u}}_k(t) \\ \underline{\tilde{u}}_k(t) &= \boldsymbol{\Phi}R\ \underline{q}_{k-1}\ \exp\ (j\omega t) \\ \underline{\tilde{f}}_k(t) &= \gamma M \underline{\tilde{u}}_k(t) - \gamma \text{ constant} \end{aligned}\right\} \tag{3-11}$$

$$\therefore \quad \underline{q}_k = \gamma \boldsymbol{\Phi}^T M \boldsymbol{\Phi} R\ \underline{q}_{k-1} \tag{3-12}$$

Using the properties of classical normal modes, we have

$$\boldsymbol{\Phi}^T M \boldsymbol{\Phi} = I \tag{3-13}$$

$$\therefore \quad \underline{q}_k = \tilde{R}\underline{q}_{k-1} \tag{3-14}$$

where

$$\left.\begin{aligned} \tilde{R}_{ii} &= \frac{\gamma \exp(-j\alpha_i)}{\sqrt{\left(\omega_i^2 - \omega^2\right)^2 + (2\omega_i\zeta_i\omega)^2}} \\ \tilde{R}_{i\ell} &= 0, \quad i \neq \ell \end{aligned}\right\} \tag{3-15}$$

Thus,

$$\left.\begin{aligned} \underline{q}_1 &= \tilde{R}\underline{q}_0 \\ \underline{q}_2 &= \tilde{R}\underline{q}_1 = \tilde{R}^2\underline{q}_0 \\ &\vdots \\ \underline{q}_n &= \tilde{R}\underline{q}_{n-1} = \tilde{R}^n\underline{q}_0 \end{aligned}\right\} \tag{3-16}$$

If

$$\frac{\left(\omega_i^2 - \omega^2\right) + (2\omega_i\zeta_i\omega)^2}{\left(\omega_\ell^2 - \omega^2\right)^2 + \left(2\omega_\ell\zeta_\ell\omega\right)^2} << 1 \quad \forall i \neq \ell \tag{3-17}$$

and if we select $\gamma \exp(-j\alpha_i)$ such that

$$\sup_{1 \leq i \leq N} \left| \tilde{R}_{ii} \right| = 1 \tag{3-18}$$

then

$$\lim_{N\to\infty} \tilde{R}^N = E_i \tag{3-19}$$

Where E_i are nonnegative, definite matrices satisfy the following property:

$$E_i E_j = E_i \delta_{ij} \tag{3-20}$$

Thus, using Equation (3-16), we have

$$\lim_{N\to\infty} \underline{q}_N = \lim_{N\to\infty} \tilde{R}^N \underline{q}_0 = q_{oi}\underline{e}_i \tag{3-21}$$

Using Equations (3-21) and (3-11), we find

$$\left.\begin{aligned} \lim_{k\to\infty} \tilde{u}_k(t) &= \lim_{k\to\infty} \mathbf{\Phi} R\underline{q}_{k-1} \exp\,(j\omega t) \\ &= \underline{\phi}^{(i)} \frac{\exp\,[j(\omega t - \alpha_i)]}{\sqrt{\left(\omega_i^2 - \omega^2\right)^2 + (2\omega_i\zeta_i\omega)^2}} \end{aligned}\right\} \tag{3-22}$$

$$\therefore \quad \left.\begin{aligned} \lim_{k\to\infty} \underline{u}_k(t) &= \mathrm{Re} \lim_{k\to\infty} \tilde{\underline{u}}(t) \\ &= \underline{\phi}^{(i)} \frac{\cos\,(\omega t - \alpha_i)}{\sqrt{\left(\omega_i^2 - \omega^2\right) + (2\omega_i\zeta_i\omega)^2}} \end{aligned}\right\} \tag{3-23}$$

Thus, the iterative process converges to a solution proportional to the pure mode $\underline{\phi}^{(i)}$, from which the natural frequency ω_i and the damping parameter ζ_i may be obtained in addition to $\underline{\phi}^{(i)}$.

It should be noted in passing that what has been said for the discrete System (2-8) is also true of the continuous System (2-1). Any system, continuous or discrete, that exhibits classical normal modes lends itself to an iterative scheme, like that above, which converges to a pure mode.

B. ERRORS IN MODAL TESTING

1. Nonclassical Normal Modes

Even if the system under test can be regarded as an N-degree-of-freedom discrete system it is highly unlikely, in practice, that the system will possess classical normal modes. Thus, theoretically, there exists no choice of forcing functions that excite pure normal modes. As previously pointed out, if the system damping is small and the eigenvalues well separated, relatively pure modes of vibration may be excited. If the system damping is not small and the eigenvalues closely spaced, as often happens in real structures, it may be impossible in practice to excite even relatively pure modes of vibrations in all but the very lowest modes.

2. Impure Modal Excitation

Even if the system under test can be regarded as an N-degree-of-freedom discrete system possessing classical normal modes, excitation of a pure mode of vibration requires that each mass in the system be excited by a force proportional to that mass and to the modal displacement of that mass. In Equation (3-1), we have shown that an iterative technique can be used to achieve this end, provided the mass matrix of the system is known and provided that we have the means to apply forces to each mass. While we will seldom know the mass matrix precisely, we often have adequately good estimates; however, we seldom have N-force transducers available to conduct the test. In the case of a continuous system, modal excitation should also be continuous, and at least an adequate discrete approximation to a continuous distribution of forces — an even more difficult task to accomplish.

3. Measurement Errors

Assuming, as in Subsections III-B-1 and III-B-2, that the discrete N degree of freedom is a good model of the system, there still remains the problem of measuring the displacements at N points for each frequency $\underline{\omega}$ and each choice of the forcing function. First of all, it is not usual to have

N displacement transducers available for a test; true, it is possible to use $N_0 < N$ transducers, and move them around the structure, but this is a time consuming procedure and greatly increases the cost of the test. However, even if we have N transducers available, and even if we could excite pure normal modes, there still exists the question of measurement error, particularly errors in phase measurements, which are notoriously difficult to make with accuracy.

4. Effects of Discretization or Condensation

Aerospace structures are almost always continuous in nature, or, at best, very-high-order discrete systems; however, for purposes of analysis and testing, we must discretize the continuous structure or condense the high-order discrete system to obtain a manageable system. In Subsection III-B it was shown that if the original system, continuous or discrete, exhibited classical normal modes, it was possible to construct an N-degree discrete model whose eigenvalue coincided exactly with the first N eigenvalues of the original system and whose eigenvectors were projections of the eigenfunctions or eigenvectors of the original system. It was further shown that for certain classes of excitation, the response of the model exactly mimicked that of the original structure. Despite these very useful properties, it should be clear that the model is not one to one with the original structure. This fact shows up immediately in modal testing. Let us suppose that we have a continuous structure and that we assign to it N coordinates $\underline{X}_i$ $i \in (1, N)$, and that we shall make measurements and apply forces only at these N points.

In Equation (2-43), let $\underline{\tilde{f}}(t)$ be given by

$$\underline{\tilde{f}}(t) = \begin{Bmatrix} \tilde{f}_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \tilde{f}_N(t) \end{Bmatrix} \tag{3-24}$$

The points of application of these forces correspond to the points x_i i ε (1, N) of the continuous system of Equation (2-1). Thus

$$f(\underline{x},t) = \sum_{i=1}^{N} \tilde{f}_i(t)\delta(\underline{x} - \underline{x}_i) \tag{3-25}$$

$$\therefore \quad q_i(t) = \int_{\mathscr{D}} \underline{X}_i(\underline{x}) f(\underline{x},t)\, d\underline{x} \tag{3-26}$$

$$= \sum_{j=1}^{N} \underline{X}_i(\underline{x}_j)\ \tilde{f}_j(t) \tag{3-27}$$

With Equations (2-39) and (2-40), if i ε (1, N), then

$$\underline{q}(t) = \begin{Bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_r \end{Bmatrix} = \mathbf{\Phi}^T \tilde{\underline{f}}(t) \tag{3-28}$$

but

$$\tilde{\underline{f}}(t) = M\mathbf{\Phi}\tilde{q}(t) \tag{3-29}$$

Thus

$$\underline{q}(t) = \mathbf{\Phi}^T\tilde{\underline{f}}(t) = \mathbf{\Phi}^T M\mathbf{\Phi}\tilde{\underline{q}}(t) \tag{3-30}$$

$$\therefore \quad \underline{q}(t) = \tilde{\underline{q}}(t)$$

provided that

$$i \ \varepsilon \ (1, N)$$

We note that

$$\underline{q}_\ell(t) = \sum_{j=1}^{N} X_\ell(x_j)\tilde{f}_j(t) \tag{3-31}$$

$$\ell \in [N + 1, \infty]$$

In general, (3-32)

$$q_\ell(t) \neq 0 \tag{3-32}$$

For example, if

$$\underline{q}(t) = q_k(t)\underline{e}_k$$

$$k \in [1, N] \tag{3-33}$$

Thus $q_\ell(t) = \tilde{q}_\ell(t)\delta_{\ell k}$ $\ell, k \in [1, N]$, but

$$q_\ell(t) = \sum_{j=1}^{N} X_\ell(\underline{x}_j)\tilde{f}_j(t) \qquad \ell > N \tag{3-34}$$

$$\neq 0 \text{ in general}$$

Thus, for the first N modes, only the k^{th} mode is excited. However, there exist higher-order modes, $\ell > N$, which are excited. If k is much smaller than N, then the separation in eigenvalues is usually such that the response of these "aliased" modes is small compared to the response of the k^{th} mode, particularly if the frequency of excitation is close to the natural frequency of the k^{th} mode. If, however, the frequency of excitation is close to that of one of the "aliased" modes, serious errors can result.

C. OTHER IDENTIFICATION TECHNIQUES

Since modal testing as an identification technique is restricted to classically damped systems, or at least systems with small damping, it is reasonable to ask if there are other identification techniques that could do a better job. There exists a variety of identification techniques, both parameter and nonparameter; however, if one wishes to identify the mass, stiffness, and damping matrices, one is faced with a fundamental limitation: the number of points N_2 at which measurements are made must, in general, be equal to N_1, the number of degrees of freedom of the structure. Unless this is done, the solutions obtained are not unique. Since, as already pointed out, aerospace structures are almost always continuous, or at least have a large number of degrees of freedom, unique identification of the structure is virtually impossible. One has to conclude therefore that, at least for structures with small damping, modal testing is probably as good an identification technique as any available.

SECTION IV
CONCLUSIONS AND RECOMMENDATIONS

The object of this report has been to examine the problems of analyses and testing of aerospace structures and the difficulties of correlating the results.

A. CONCLUSIONS

The following conclusions have emerged from this study:

(1) Modal testing (and indeed any other technique) cannot be used as a method for uniquely determining the mass, damping, and stiffness matrices of real aerospace structures, which are usually continuous in nature, and the modal testing thereby cannot provide a means of improving the analytical techniques for determining dynamics response. Modal testing is an extremely useful tool for obtaining accurate measures of the eigenvalues and eigenvectors of the lower modes of the structure. These measures can be correlated with analytical results, or provide the basis for discrete models of the structure, which may be extremely useful in the analysis of stability and control.

(2) Analytical techniques are capable of modeling dynamic structures to any desired degree of accuracy. It is clear that eigenvalues can be predicted with a higher degree of accuracy than can eigenvectors, and that eigenvectors can be predicted with a higher degree of accuracy than that accuracy with which the modal force coefficient can be predicted. Accurate analytical prediction of stresses and forces will require a finer level of discretization than will the prediction of eigenvalues and eigenvectors. If accurate analytical predictions of stress and forces are required, the additional effort and expense of using finer and finer meshes must be accepted.

(3) Accuracy of modeling is a central question in all mathematical modeling: "Given that the data has only finite accuracy, how accurate need the model be to obtain acceptable accuracy in the response?" This topic was dealt with at some length in Subsection II-C, and the answer depends on the form of excitation. For short transients, the model need not be any more accurate than the input data; for persistent inputs that create the possibility of resonance, the model must be specified with a much higher degree of accuracy than the input data. These results are consistent with the results obtained by Chen and Wada (Reference 2-3).

B. RECOMMENDATIONS

As a result of the present study, two recommendations emerge:

(1) To determine stresses and forces in aerospace systems, an appropriate level of discretization must be used, even if this is much finer than would be used for determining eigenvalues and eigenvectors.

(2) While it is virtually impossible to "identify" the structure uniquely from the result of modal testing, such tests provide a valuable check on the analytical method and can be used to provide an accurate, discrete model of the system for use in studies of stability and control. Modal testing has an additional virtue that should be exploited to the fullest. There are many physical phenomena, such as the sloshing of fuel in a spinning spacecraft that are rather difficult to model accurately since boundary layer friction and dissipation play a central role. In such a case, modal tests of the physical system can easily provide the data on which to base an analytical model. This modal testing then becomes part and parcel of the modeling technique in which some parts of the structure are modeled ab initio, and some parts are modeled on the basis of the modal test.

REFERENCES

2-1. Caughey, T. K., and O'Kelly, M. E. J., "Classical Normal Modes in Damped Linear Dynamic Systems," J. Appl. Mech. Vol. 39, No. 3, Sept. 1965, pp. 583-588; also, Caughey, T. K., "Classical Normal Modes in Damped Linear Dynamic Systems," J. Appl. Mech. Vol. 27, No. 2, June 1960, pp. 269-271.

2-2. O'Kelly, M. E. J., Normal Modes in Damped System, Engineer's Thesis, Caltech 1961.

2-3. Chen, J. C., and Wada, B. K., "Criteria for Analysis - Test Correlation of Dynamic Structures," J. Appl. Mech. Vol. 42, No. 2, June 1975, pp. 471-477.

APPENDIX

AN EXAMPLE

Consider the problem

$$\frac{\partial^2 u}{\partial t^2} + 2\beta \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(\underline{x}, t) \qquad 0 < x < 1$$

$$u(0, t) = u(1, t) = 0 \tag{A-1}$$

$$u(x, 0) = \dot{u}(x, 0) = 0$$

The eigenvalues and eigenfunctions of Equation (A-1) are

$$\omega_i^2 = (i\pi)^2 \qquad i \,\varepsilon\, [1, \infty)$$

$$X^{(i)}(x) = \sqrt{2} \sin i\pi x \tag{A-2}$$

If we write

$$u(\underline{x}, t) = \sum_{i=1}^{\infty} Z_i(t) X^{(i)}(x) \tag{A-3}$$

then

$$\ddot{Z}_i + 2\beta \dot{Z}_i + \omega_i^2 Z_i = q_i(t) \tag{A-4}$$

where

$$u_i(0) = \dot{u}_i(0) = 0$$

and

$$q_i(t) = \int_0^1 f(x, t)\; x^{(i)}(x)\; dx \tag{A-5}$$

Discretization of Equation (A-1)

If we use central difference spatial discretization or constant mass matrix finite element spatial discretization, Equation (A-1) becomes

$$\ddot{\tilde{u}}_i + 2\beta\dot{\tilde{u}}_i + (N)^2 \left[2\tilde{u}_i - \tilde{u}_{i+1} - \tilde{u}_{i-1}\right] = \tilde{f}_i(t) \tag{A-6}$$

where

$$\tilde{u}_i(t) = u(ih,\ t) \qquad \tilde{f}_i(t) = f(ih,\ t) \tag{A-7}$$

$$h = \frac{1}{N},\quad i\ \varepsilon \left[1,\ (N-1)\right]$$

The eigenvalues and eigenvectors of Equation (A-6) are

$$\left.\begin{aligned} \Omega_i^2 &= 4N^2 \sin^2 \frac{i\pi}{2N} \\ \phi_j^{(i)} &= \sqrt{\frac{2}{N}} \sin \frac{ij\pi}{N} \end{aligned}\right\} \tag{A-8}$$

Let

$$\left.\begin{aligned} \underline{\phi}^{(i)} &= \begin{Bmatrix} \phi_1^{(i)} \\ \phi_2^{(i)} \\ \cdot \\ \cdot \\ \cdot \\ \phi_{N-1}^{(i)} \end{Bmatrix} \qquad \mathbf{\Phi} = \left[\underline{\phi}^{(1)},\ \underline{\phi}^{(2)},\ \cdots\ \underline{\phi}^{(N-1)}\right] \\ \underline{\tilde{u}} &= \begin{Bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_{N-1} \end{Bmatrix} \end{aligned}\right\} \tag{A-9}$$

Let

$$\left.\begin{aligned} \underline{u} &= \boldsymbol{\Phi}\, \underline{\tilde{Z}} \\ \underline{\tilde{Z}} &= \begin{Bmatrix} \tilde{Z}_1 \\ \tilde{Z}_2 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{Z}_{N-1} \end{Bmatrix} \end{aligned}\right\} \qquad \text{(A-10)}$$

where

$$\therefore \quad \ddot{\tilde{Z}}_i + 2\beta\dot{\tilde{Z}}_i + \Omega_i^2\, \tilde{Z}_i = \tilde{q}_i(t) \qquad \text{(A-11)}$$

$$\underline{\tilde{q}}(t) = \begin{Bmatrix} \tilde{q}_1 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{q}_{N-1} \end{Bmatrix} = \boldsymbol{\Phi}\, \tilde{f}(t)$$

$$\tilde{q}_i(t) = \sum_{j=1}^{N-1} \phi_i^{(j)}\, \tilde{f}_j(t)$$

Comparisons of Equations (A-8) and (A-9) shows that

$$\phi_j^{(i)} \propto X^{(i)}\left(\frac{j}{N}\right) \qquad \text{(A-12)}$$

$$\left|\frac{\Omega_i - \omega_i}{\omega_i}\right| = \left|\frac{2N \sin\frac{i\pi}{2N}}{i\pi} - 1\right| = \left|\frac{\sin\frac{i\pi}{2N}}{\left(\frac{i\pi}{2N}\right)} - 1\right| \qquad \text{(A-13)}$$

$$i \;\varepsilon \left[1,\; N - 1\right]$$

Thus, in this case, the eigenvectors of the approximating system are projections of the first N eigenfunctions of the continuous system.

For the sake of illustration, let us use N = 4. Equation (A-13) then gives

$$\varepsilon_i = \left| \frac{\Omega_i - \omega_i}{\omega_i} \right| = \left| \frac{\sin \frac{i\pi}{8}}{\frac{i\pi}{8}} - 1 \right|, \quad i \in \left[1, 3\right] \tag{A-14}$$

From Equation (A-14)

$$\left.\begin{aligned} \varepsilon_1 &= 0.0255 \\ \varepsilon_2 &= 0.0997 \\ \varepsilon_3 &= 0.2158 \end{aligned}\right\} \tag{A-15}$$

This shows clearly how the errors increase with mode order. If Equation (A-6) is written in matrix form,

$$\underline{M}\ddot{\tilde{u}} + 2\beta\underline{M}\dot{\tilde{u}} + \underline{K}\tilde{u} = \underline{g}(t) \tag{A-16}$$

$$\left.\begin{aligned} M &= \frac{1}{4} I \\ K &= 4 \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \\ \underline{g}(t) &= \frac{1}{4} \begin{Bmatrix} f\left(\frac{1}{N}, t\right) \\ f\left(\frac{2}{N}, t\right) \\ f\left(\frac{3}{N}, t\right) \end{Bmatrix} \end{aligned}\right\} \tag{A-17}$$

Let us now use Theorem 1 to construct a 3^{rd}-order system having the same eigenvalues of the continuous system. Let us select

$$x_j = \frac{j}{4} \qquad j \in \left[1,\ 3\right] \tag{A-18}$$

Thus

$$\phi^{(i)} = \begin{pmatrix} \sqrt{2}\,\sin\frac{j\pi}{4} \\ \sqrt{2}\,\sin\frac{2j\pi}{4} \\ \sqrt{2}\,\sin\frac{3j\pi}{4} \end{pmatrix} \tag{A-19}$$

Hence

$$\boldsymbol{\Phi} = \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} \tag{A-20}$$

Using Equation (2-41), we have

$$\begin{aligned} M_0 &= \frac{1}{4}\, I \\ D_0 &= \frac{\beta}{2}\, I \\ K_0 &= \frac{\pi^2}{16} \begin{bmatrix} 18 & -8\sqrt{2} & 2 \\ -8\sqrt{2} & 20 & -8\sqrt{2} \\ 2 & -8\sqrt{2} & 18 \end{bmatrix} \end{aligned} \tag{A-21}$$

In this case we have chosen α^2 so that M is the same as that obtained by finite difference. Thus,

$$K_0 = \begin{bmatrix} 11.103 & -6.9785 & 1.2337 \\ -6.9785 & 12.3370 & -6.9785 \\ 1.2337 & -6.9785 & 11.103 \end{bmatrix} \qquad \text{(A-22)}$$

and

$$K = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} \qquad \text{(A-23)}$$

Comparison of Equations (A-17) and (A-21) show that both models have the same M and D matrices (they were so constructed); however, the K matrices of Equations (A-23) and (A-27) show considerable differences:

(1) K_0 is a full matrix, while K is a tridiagonal Jacobi matrix.

(2) The magnitudes of the elements of the two K matrices are markedly different.

Thus, we see that if modal testing is used to identify a 3^{rd}-order model of the continuous system, the matrices of Equation (A-17) would result. If the stiffness matrix K_0 is compared with K, the matrix obtained from finite difference or finite element discretization, we see that they're not even close. Hence, we see that a stiffness matrix obtained from modal testing <u>cannot</u> be used to check that obtained from systematic analytic reduction techniques such as finite difference or finite element.

To illustrate aliasing, suppose that g(t) in Equation (A-16) is given by

$$\underline{g}(t) = \sqrt{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \cos \omega t \tag{A-24}$$

which will excite only the first mode of oscillation of Equation (A-16).

Using Equations (3-25), (3-26), and (3-27), we have

$$q_i(t) = \sqrt{2} \left[\sin \frac{i\pi}{4} + \sqrt{2} \sin \frac{2i\pi}{4} + \sin \frac{3i\pi}{4} \right] \cos \omega t \tag{A-25}$$

$$= 2 \sin \frac{i\pi}{2} \left[\sqrt{2} \cos \frac{i\pi}{4} + 1 \right] \cos \omega t \tag{A-26}$$

Thus

$$q_i(t) \equiv 0 \text{ unless } i = 8k \pm 1 \tag{A-27}$$

$$k = 0, 1, 2 \cdots$$

$$q_{8k\pm 1}(t) = (\pm 1)\ 4 \cos \omega t \tag{A-28}$$

Thus

$$q_1(t),\ q_7(t),\ q_9(t) \text{ etc. } \neq 0$$

If $0 \leq \omega < 4\pi$, only the first mode will be strongly excited. If, however, $0 \leq \omega < 8\pi$, the 1^{st} and 7^{th} mode of the continuous structure can be strongly excited.

For structures with well separated eigenvalues and excitation restricted to the bandwidth of the first N modes, aliasing does not present a serious problem. Some structures, such as shell-like structures, tend to have rather closely spaced eigenvalues and, in this case, aliasing becomes a more serious problem. While this discussion was restricted for simplicity to the case $N = 4$, the same features show up for all values of N.